

Some Lessons from the Schwinger Model

Gary McCartor

Department of Physics, Southern Methodist University, Dallas, TX 75275

(September 17, 1996)

Abstract

I shall recall a number of solutions to the Schwinger model in different gauges, having different boundary conditions and using different quantization surfaces. I shall discuss various properties of these solutions emphasizing the degrees of freedom necessary to represent the solution, the way the operator products are defined and the effects these features have on the chiral condensate.

I. INTRODUCTION

In this talk I shall discuss the following solutions to the Schwinger model:

1. The solution of Lowenstein and Swieca¹ in Lorentz gauge and in the continuum (that is, no periodicity conditions imposed). That solution has a chiral condensate given by:

$$\langle \Omega | \bar{\psi} \psi | \Omega \rangle = \frac{m}{2\pi} e^{\gamma \cos \theta}$$

2. The solution of Nakawaki² in Coulomb gauge and with antiperiodic boundary conditions on the surface $t = 0$. That solution has a chiral condensate given by:

$$\langle \Omega | \bar{\psi} \psi | \Omega \rangle = \frac{1}{2L} e^X \cos \theta,$$

which in the large L limit goes to:

$$\langle \Omega | \bar{\psi} \psi | \Omega \rangle = \frac{m}{2\pi} e^{\gamma \cos \theta}$$

3. The solution I gave³ in light-cone ($\partial_- A^+ = 0$) gauge with antiperiodic boundary conditions on $x^+ = 0$ (for ψ_+) and antiperiodic boundary conditions on $x^- = 0$ (for ψ_-). That solution has a chiral condensate given by:

$$\langle \Omega | \bar{\psi} \psi | \Omega \rangle = \frac{1}{L} \cos \theta,$$

which in the large L limit goes to 0.

4. The solution of Nakawaki⁴ in light-cone gauge in the continuum. That solution has a chiral condensate given by:

$$\langle \Omega | \bar{\psi} \psi | \Omega \rangle = \frac{m}{2\pi} e^{\gamma \cos \theta}$$

5. The solution of Vianello⁵ in light-cone ($A^+ = 0$) gauge with antiperiodic boundary conditions on the surface $t = 0$. That solution has a chiral condensate given by:

$$\langle \Omega | \bar{\psi} \psi | \Omega \rangle = \frac{1}{2L} e^X \cos \theta,$$

which in the large L limit goes to:

$$\langle \Omega | \bar{\psi} \psi | \Omega \rangle = \frac{m}{2\pi} e^{\gamma \cos \theta}$$

A principal focus of our discussion will be how operator products are regulated in these solutions and the effect of that regulation on the chiral condensate.

II. CHIRAL CONDENSATE

Let us first review, in some detail, how the chiral condensate arises. We shall consider Nakawaki's² Coulomb gauge solution—that is probably the most straightforward case.

To produce that solution the ψ - fields are initialized on $t = 0$ as isomorphic to free, massless Fermi fields:

$$\begin{aligned}\psi_1(x) &= \frac{1}{\sqrt{2L}} e^{-\lambda_1^{(-)}(x)} \sigma_1(x) e^{-\lambda_1^{(+)}(x)} \\ \psi_2(x) &= \frac{1}{\sqrt{2L}} e^{-\lambda_2^{(-)}(x)} \sigma_2(x) e^{-\lambda_2^{(+)}(x)}\end{aligned}$$

Where:

$$\begin{aligned}\lambda_1^{(+)} &= - \sum_{n=1}^{\infty} \frac{1}{n} D(n) e^{-iq(n)x} \quad ; \quad \lambda_1^{(-)} = \sum_{n=1}^{\infty} \frac{1}{n} D^*(n) e^{iq(n)x} \\ \lambda_2^{(+)} &= - \sum_{n=1}^{\infty} \frac{1}{n} C(n) e^{-iq(-n)x} \quad ; \quad \lambda_2^{(-)} = \sum_{n=1}^{\infty} \frac{1}{n} C^*(n) e^{iq(-n)x},\end{aligned}$$

the C 's and D 's are the fusion operators associated with Bosonizing the free massless Fermi field, the σ 's are spurion operators, and $q(n) = \frac{n\pi}{L}$. Of interest later will be the states which are destroyed by the positive frequency fusion operators. Defining:

$$\begin{aligned}|M, N\rangle &= \delta^*(M) \dots \delta^*(1) d^*(N) \dots d^*(1) |0\rangle \quad (M > 0, N > 0) \\ |M, N\rangle &= \beta^*(M) \dots \beta^*(1) d^*(N) \dots d^*(1) |0\rangle \quad (M < 0, N > 0) \\ |M, N\rangle &= \delta^*(M) \dots \delta^*(1) b^*(N) \dots b^*(1) |0\rangle \quad (M > 0, N < 0) \\ |M, N\rangle &= \beta^*(M) \dots \beta^*(1) b^*(N) \dots b^*(1) |0\rangle \quad (M < 0, N < 0)\end{aligned}$$

we have:

$$(C \text{ or } D)|M, N\rangle = 0$$

and no other states have this property. In this gauge, with these boundary conditions, there is a single degree of freedom in the field A^1 (independent of x^1). It is convenient to define:

$$A_1 = \frac{1}{\sqrt{2\pi}} (\alpha_1 + \alpha_1^*)$$

$$\partial_0 A_1 = \frac{\sqrt{\pi}}{2iL} (\alpha_1 - \alpha_1^*)$$

where the α 's satisfy the commutation relations for a single Bose mode. With these definitions we can calculate the Hamiltonian:

$$\begin{aligned}H &= \frac{\pi}{4Le^2} (Q^2 + Q_5^2) + \frac{\pi}{L} \sum_{n=1}^{\infty} (C^*(n)C(n) + D^*(n)D(n)) \\ &+ \frac{e^2}{4L} \sum_{n=1}^{\infty} \frac{1}{q_1^2(n)} :[D(n) + C^*(n)][C(n) + D^*(n)] + [C(n) + D^*(n)][D(n) + C^*(n)] : \\ &+ :L(\partial_0 A^1)^2 - A_1 Q_5 + \frac{e^2}{\pi} L A_1^2 : \end{aligned}$$

where Q and Q_5 are respectively the charge and the pseudocharge.

To diagonalize the Hamiltonian we define a set of operators $a(n)$ implicitly through the relations ($n \neq 0$):

$$C(n) = -i\sqrt{n}(\cosh(\theta(n))a(-n) - \sinh(\theta(n))a^*(n))$$

$$C^*(n) = i\sqrt{n}(\cosh(\theta(n))a^*(-n) - \sinh(\theta(n))a(n))$$

$$D(n) = -i\sqrt{n}(\cosh(\theta(n))a(n) - \sinh(\theta(n))a^*(-n))$$

$$D^*(n) = i\sqrt{n}(\cosh(\theta(n))a^*(n) - \sinh(\theta(n))a(-n))$$

Here we use the c-number functions:

$$P_1 = \frac{n\pi}{L} \quad P_0(n) = \sqrt{m^2 + P_1^2(n)} \quad \theta(n) = \frac{1}{2} \ln \frac{P_0(n)}{|P_1(n)|}$$

For $n = 0$ we have:

$$a(0) \equiv i \left(\sqrt{\frac{eL}{\sqrt{\pi}}} A_1 + i \sqrt{\frac{\sqrt{\pi}L}{e}} \partial_0 A_1 - \frac{\pi^{\frac{3}{4}}}{2e^{\frac{3}{2}}\sqrt{L}} Q_5 \right)$$

$$a^*(0) \equiv -i \left(\sqrt{\frac{eL}{\sqrt{\pi}}} A_1 - i \sqrt{\frac{\sqrt{\pi}L}{e}} \partial_0 A_1 - \frac{\pi^{\frac{3}{4}}}{2e^{\frac{3}{2}}\sqrt{L}} Q_5 \right)$$

In the new variables the Hamiltonian is:

$$H = \frac{\pi}{4Le^2} Q^2 + \sum_{n=-\infty}^{\infty} P^0(n) a^*(n) a(n)$$

From which we see that $|\Omega\rangle$ will be a ground state of the system if:

$$a(n)|\Omega\rangle = 0$$

and:

$$Q|\Omega\rangle = 0$$

This last requirement is necessary for all states in the physical subspace.

The similarity transformation which diagonalizes the Hamiltonian is:

$$SD(n)S^{-1} = i\sqrt{n}a(n) \quad n > 0$$

$$SC(-n)S^{-1} = i\sqrt{n}a(n) \quad n < 0$$

$$S\alpha_1 S^{-1} = a(0)$$

$$S = S_0 S'$$

$$S_0 = \exp \left[\frac{\theta(0)}{2} (\alpha_1^2 - \alpha_1^{*2}) - \frac{\theta(0)}{e^{\theta(0)} - 1} \frac{Q_5}{2m\sqrt{Lm}} (\alpha_1 - \alpha_1^*) \right]$$

$$S' = \exp\left[\sum_{n=1}^{\infty} \frac{\theta(n)}{n} (C(n)D(n) - D^*(n)C^*(n))\right]$$

Since:

$$a(n)S|M, N\rangle = S(CorD)S^{-1}S|M, N\rangle = 0$$

and:

$$SQS^{-1} = Q$$

We see that any state of the form:

$$|\Omega(M)\rangle \equiv S|M, -M\rangle$$

will be a ground state of the system. It is easy to show that:

$$\langle\Omega(M)|\bar{\psi}\psi|\Omega(M)\rangle = 0$$

So if $|\Omega(M)\rangle$ is chosen for the ground state there is no chiral condensate. There is a subtle reason why that is not a good choice however⁶. If we wish to have a solution which satisfies the cluster property we must have the vacuum be an eigenstate of the mass operator:

$$(\psi_1^*\psi_2 + \psi_2^*\psi_1)|\Omega\rangle \sim |\Omega\rangle$$

To do that we must choose the vacuum to be a θ -state:

$$|\Omega(\theta)\rangle \equiv \sum_{n=-\infty}^{\infty} e^{iM\theta} |\Omega(M)\rangle$$

We can now calculate the chiral condensate. We rewrite the fields in terms of the new variables:

$$\psi_1^*\psi_2 = \frac{1}{2L}\underline{\sigma}_1^*\underline{\sigma}_2 e^{\lambda_1^{(-)}} e^{\lambda_1^{(+)}} e^{-\lambda_2^{(-)}} e^{-\lambda_2^{(+)}}$$

Where:

$$\begin{aligned}\lambda_1^{(+)} &= \sum_{n=1}^{\infty} \frac{i}{\sqrt{n}} (\cosh(\theta(n))a(n)e^{-ipx} - \sinh(\theta(n))a^*(-n))e^{ipx} \\ \lambda_1^{(-)} &= \sum_{n=1}^{\infty} \frac{i}{\sqrt{n}} (\cosh(\theta(n))a^*(n)e^{ipx} - \sinh(\theta(n))a(-n))e^{-ipx} \\ -\lambda_2^{(+)} &= \sum_{n=1}^{\infty} \frac{-i}{\sqrt{n}} (\cosh(\theta(n))a(-n)e^{-ipx} - \sinh(\theta(n))a^*(n)e^{ipx}) \\ -\lambda_2^{(-)} &= \sum_{n=1}^{\infty} \frac{-i}{\sqrt{n}} (\cosh(\theta(n))a^*(-n)e^{ipx} - \sinh(\theta(n))a(n))e^{-ipx}\end{aligned}$$

If we now commute all the destruction operators forward and all the creation operators backward, and use the relations:

$$\underline{\sigma}_1^*\underline{\sigma}_2|\Omega(M)\rangle = |\Omega(M+1)\rangle$$

$$\underline{g}_1^* \underline{g}_2 |\Omega(\theta)\rangle = e^{-i\theta} |\Omega(\theta)\rangle$$

we find that:

$$\psi_1^* \psi_2 |\Omega(\theta)\rangle = \frac{1}{2L} e^X e^{-i\theta} |\Omega(\theta)\rangle$$

Where:

$$e^X = \exp \left[-2 \sum_{n=1}^{\infty} \frac{1}{n} (\sinh^2(\theta(n)) - \sinh(\theta(n)) \cosh(\theta(n))) \right]$$

It is easy to show that:

$$\lim_{L \rightarrow \infty} e^X = \frac{m}{2\pi} e^\gamma L$$

So we find that:

$$\lim_{L \rightarrow \infty} \langle \Omega | \bar{\psi} \psi | \Omega \rangle = \frac{m}{2\pi} e^\gamma \cos \theta$$

We thus see that the fact that the chiral condensate survives to large L is due to the infrared divergence in the factor e^X ; that factor arises from the fact that there is mixing between the positive and negative frequencies due to the interaction—it has the same source as the wavefunction renormalization constant.

It will be important for future considerations to note that in this solution Fermi products are regularized as:

$$:\psi^\dagger \psi: = \lim_{\substack{\epsilon \rightarrow 0 \\ \epsilon^2 < 0}} \left\{ e^{-ie \int_x^{x+\epsilon} A_\nu^{(-)} dx^\nu} \psi^\dagger(x+\epsilon) \psi(x) e^{-ie \int_x^{x+\epsilon} A_\nu^{(+)} dx^\nu} - \text{V.E.V.} \right\}$$

That is, with a gauge invariant splitting which can be on the spacelike initial value surface $t = 0$; as in the case of free theory, however, a splitting in the lightlike direction x^- is allowed. We shall return to this point later.

III. ON THE LIGHT-CONE

Let us now contrast the above results with those obtained³ when the same Lagrangian is quantized on characteristic surfaces: ψ_+ on $x^+ = 0$; ψ_- on $x^- = 0$, with antiperiodic boundary conditions in each case. We use the gauge $\partial_- A^+ = 0$. We initialize the fields to be isomorphic to free fields on these surfaces:

$$\psi_+ = \frac{1}{\sqrt{2L}} e^{\lambda_+^{(-)}(x)} \sigma_+(x) e^{\lambda_+^{(+)}(x)}$$

$$\psi_- = \frac{1}{\sqrt{2L}} e^{-\lambda_D^*(x^+)} \sigma_-(x) e^{\lambda_D(x^+)}$$

where:

$$\lambda_+(x) = -i \sqrt{\frac{\pi}{L}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{p_-(n)}} \left(C(n) e^{-ip(n)x} + C^*(n) e^{ip(n)x} \right)$$

$$\lambda_D(x^+) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} D(n) e^{-ik_+(n)x^+}$$

In this case there is a zero mode in each component of the gauge field but neither is a degree of freedom. These zero modes are found to be:

$$A^+ = -\frac{1}{Lm^2}Q_- \quad ; \quad A^-(0) = -\frac{1}{Lm^2}Q_+$$

The operator which controls the dynamics, P^- , is found to be:

$$P^- = \frac{1}{4Lm^2}(Q_-^2 - Q_+^2) + \sum_{n=1}^{\infty} p^-(n)C^*(n)C(n) + \sum_{n=1}^{\infty} 2k_+(n)D^*(n)D(n)$$

where Q_+ and Q_- are respectively the charge in the ψ_+ or ψ_- field and we have used the c-number functions:

$$P^-(n) = \frac{m^2 L}{2n\pi} \quad ; \quad k_+(n) = \frac{n\pi}{L}$$

Thus the Hamiltonian and the momentum are already diagonal and there is no need for an S-operator.

Arguments essentially similar to those we used in Coulomb gauge tell us that if we define:

$$|\Omega(M)\rangle = |M, -M\rangle$$

the ground state of the system will be:

$$|\Omega(\theta)\rangle \equiv \sum_{n=-\infty}^{\infty} e^{iM\theta} |\Omega(M)\rangle$$

An easy calculation gives:

$$\psi_+^* \psi_- |\Omega(\theta)\rangle = \frac{1}{2L} e^{-i\theta} |\Omega(\theta)\rangle$$

so the chiral condensate is:

$$\langle \Omega | \bar{\psi} \psi | \Omega \rangle = \frac{1}{L} \cos \theta,$$

which in the large L limit goes to 0. The reason the chiral condensate vanishes in the large L limit, in contrast to the previous section, is the lack of the factor e^X . That factor arose due to the mixing of the positive and negative frequency modes and its absence here is due to the lack of such mixing. The fact that the positive and negative frequency modes do not mix in this gauge with these boundary conditions was first discussed by Eller, Pauli and Brodsky⁷. For the exact same reason there is no wave function renormalization. Note here that we may define Fermi products as:

$$:\psi_+^*(x)\psi_+(x): \equiv \lim_{\epsilon^- \rightarrow 0} \left(e^{-ie \int_x^{x+\epsilon^-} A_-^{(-)} dx^-} \psi_+^*(x + \epsilon^-) \psi_+(x) e^{-ie \int_x^{x+\epsilon^-} A_-^{(+)} dx^-} - \text{V.E.V.} \right)$$

Splittings in spacelike directions are also allowed, but the point is that, just as in free theory, splitting in the lightlike initial value surface is allowed.

We might wonder whether the differences between the two solutions we have discussed are due to the choice of gauge or the choice of boundary conditions. Let us briefly consider the solution⁵ in light-cone gauge initialized on the surface $t = 0$ with antiperiodic boundary

conditions on the Fermi fields and periodic boundary conditions on the gauge fields (the field A^- is no longer a constraint). The Fermi fields are initialized as in the case of the Coulomb gauge and again diagonalization of the Hamiltonian requires mixing of the negative and positive frequency modes. The diagonalization is accomplished in terms of a massive pseudoscalar field, $\tilde{\Sigma}$, half the degrees of freedom of a massless ghost, $\tilde{\eta}$ and half the degrees of freedom of a massless scalar, λ_1 :

$$\psi_- = \psi_-(FREE) = \frac{1}{\sqrt{2L}} e^{-\lambda_1^{(-)}(x)} \sigma_1(x) e^{-\lambda_1^{(+)}(x)}$$

$$\psi_+ = e^{[-2i\sqrt{\pi}(\tilde{\eta}^{(-)} + \tilde{\Sigma}^{(-)})]} \sigma_+ e^{[-2i\sqrt{\pi}(\tilde{\eta}^{(+)} + \tilde{\Sigma}^{(+)})]}$$

The ghost field plays an essential role in regulating Fermi products. We have:

$$\langle e^{[-2i\sqrt{\pi}\tilde{\Sigma}^{(+)}(x+\epsilon)]} e^{[-2i\sqrt{\pi}\tilde{\Sigma}^{(-)}(x)]} \rangle \approx e^X \frac{-i}{2\pi\epsilon^-} \frac{-i}{2\pi\epsilon^+}$$

while:

$$\langle e^{[-2i\sqrt{\pi}\tilde{\eta}^{(+)}(x+\epsilon)]} e^{[-2i\sqrt{\pi}\tilde{\eta}^{(-)}(x)]} \rangle \approx \left(\frac{-i}{2\pi\epsilon^+}\right)^{-1}$$

We thus find that the proper working of the gauge invariant point splitting regulator requires that we split in a spacelike direction and have the ghost field present. The x^+ singularity in the first case comes from the small P^- region of momentum space, while in the second case it comes from the large P^- region. The coulomb gauge solution split the singularities in x^- and x^+ between the ψ_+ and ψ_- fields just as in free theory, but in light-cone gauge all the dynamics is placed on the ψ_+ field and both singularities arise, one of them canceled by the ghost. In light-cone gauge with antiperiodic boundary conditions on $x^+ = 0$, the small P^- region of momentum space is removed and the x^+ singularity does not arise.

The solution⁵ just discussed goes smoothly into the light-cone gauge continuum solution of Nakawaki⁴. In the continuum case the solution is also written in terms of a massive pseudoscalar, a massless ghost and a massless scalar. In fact one can almost get from the periodic solution to the continuum one by changing the dispersion relation for all fields in the periodic solution to the relevant continuum one. The extra differences are that a more complicated spurion is needed in the continuum case and a Klaiber⁸ regulator is needed to control the infrared. One still finds that the splitting must be in a strictly spacelike direction with the ghost field cancelling out an unwanted singularity just as in the periodic case. That means that while Nakawaki's solution can be quantized on either $t = 0$ or $x^+ = 0$, if the characteristic surface is used then dynamics gets involved in the definition of the operators and there arises extra subtleties.

IV. REMARKS

All the solutions in light-cone gauge have fields which are functions of x^+ and which are essential parts of the solution. Thus, in that gauge one must use either a spacelike surface or both characteristic surfaces, x^+ and x^- , to properly initialize the problem.

The light-cone gauge solutions which have:

$$\langle \Omega | \bar{\psi} \psi | \Omega \rangle \neq 0$$

in the limit as the regulator is removed have the property that point splitting must be done on a spacelike surface. This fact gives rise to extra complications in quantizing these solutions on the characteristic surfaces in that the definition of the operator products must involve points outside the initial value surface and thus dynamics gets involved in defining the dynamical generators. It is not clear how this complication carries over to higher dimensions. The functions of x^+ are present in gauge theories in all dimensions but a nonperturbative regulator for defining field products, with a procedure for renormalizing these products is not known. Thus, even though the field products could be split in a spacelike direction within the initial value surface for theories in dimensions higher than two, the implications of this fact for a possible kinematical definition of the field products is not clear.

If we impose periodicity conditions on the characteristic surfaces, the operator products can be regulated by splitting in the initial value surface. This greatly simplifies the technical formulation of the theory but has the property that some of the subtle details of the operator products, such as the condensate, do not approach their continuum values as the periodicity length is taken to infinity. However, the light-cone gauge solution with periodicity conditions on the characteristics gives the correct spectrum and the correct s-matrix while having a much simpler (though not completely trivial) vacuum and a much simpler solution. The anomaly is also correctly given. The degree to which one may lose the ability to represent some aspects of the physics in order to obtain the simplifications derived from a kinematic definition of the field products is not entirely understood. It would be very valuable to be able to make more general statements on this subject.

The procedure of introducing periodicity conditions on the characteristic surfaces is very similar to the regulator used by 't Hooft⁹ to solve large-N QCD in two dimensions. Both have the effect of simply removing the small P^+ region without inducing any counterterms. That solution is known to have an inconsistency involving the condensate: The 't Hooft⁹ propagator has zero condensate but the spectrum derived from it implies a nonzero value for the condensate¹⁰. I believe that the features of the Schwinger model solutions we have been discussing will be found relevant to resolving that apparent inconsistency in the solution to the 't Hooft model.

REFERENCES

1. J. H. Lowenstein and J. A. Swieca, Ann. of Phys. **68**, 172 (1971).
2. Y. Nakawaki, Prog. Theor. Phys. **70**, 1105 (1983).
3. G. McCartor, Z. Phys. **C64**, 349 (1994)
4. Y. Nakawaki, Private Communication
5. E. Vianello, Private Communication
6. J. Kogut and L. Susskind, Phys. Rev. **D11**, 3594 (1975)
7. T. Eller, H.-C. Pauli and S. J. Brodsky, Phys. Rev. **D35**, 1493 (1987)
8. B. Klaiber, Boulder Lectures in Theoretical Physics (Gordon and Breach,1697), vol.XA
9. G. 't Hooft, Nucl. Phys. **B75**, 461 (1974)
10. A. Zhitnitsky, Phys. Lett. **B165**, 405 (1985)